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Schrödinger operators with nonlocal point interactions

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Abstract

Schrödinger operators with nonlocal point interactions are considered as new solvable models with point interactions. Examples in one and three dimensions are discussed. Corresponding direct and inverse scattering problems in one dimension are also discussed.

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1. Introduction

It is well known [2,3,9,20] that the Schrödinger operator $-\Delta_{\alpha,0}$ with a one-point interaction in the point $x = 0$ is self-adjoint on the space $L_2(\mathbb{R}^3)$ and is defined by the expression $-\Delta$ on complex-valued functions ψ that have a singularity at zero of the type $|x|^{-1}$,

$$\psi(x) = \frac{\psi_s}{4\pi|x|} + \psi_r + o(1), \quad |x| \rightarrow 0, \quad (1)$$

with ψ_s, ψ_r being complex constants.

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The domain of the operator $-\Delta_{\alpha,0}$ is defined by the condition

$$\psi_s = \alpha \psi_r, \quad (2)$$

where the real parameter α , which has the physical meaning of the inverse of the scattering length, fixes the intensity of the point interaction.

Considering the Schrödinger operator $L = -\Delta + v$ with a potential v that has a small interaction radius, $R = \int |xv(x)| dx / \int |v(x)| dx \ll 1$, one might expect that the point interaction operator $-\Delta_{\alpha,0}$ models well the operator L . This means that the operator L must be considered on functions ψ satisfying condition (1). For potentials with a small interaction radius, one can take the approximation $v(x)\psi(x) = v(x)(\frac{\psi_s}{4\pi|x|} + \psi_r)$ in the Schrödinger equation $-\Delta\psi + v\psi = E\psi$ in the case where the energy E is small. Thus it is necessary to consider Schrödinger operators of the form

$$L_f\psi = -\Delta\psi(x) + f_1(x)\psi_s + f_2(x)\psi_r, \quad x \neq 0, \quad (3)$$

with

$$\psi_s = 4\pi \lim_{|x| \rightarrow 0} |x|\psi(x), \quad \psi_r = \lim_{|x| \rightarrow 0} \left(\psi(x) - \frac{\psi_s}{4\pi|x|} \right), \quad (4)$$

and a given pair of complex-valued functions, $(f_1, f_2) \equiv f$.

The Schrödinger operator $L\psi = -\Delta\psi(x) + \int K(x, s)\psi(s)ds$, with a nonlocal interaction given by K , can formally be reduced to an operator of the form (3) if the kernel K can be written as $K(x, s) = f(x)\delta(s) + \delta(x)\overline{f(s)}$, where δ is Dirac's δ -function, so that the Schrödinger operator has the form

$$L_f = -\Delta + f(\cdot, \delta) + \delta(\cdot, f). \quad (5)$$

In the case where an operator of type (5) is considered acting on functions ψ of the form (1) and taking $(\psi, \delta) = \psi_r$ on such functions, the condition $L\psi \in L_2(\mathbb{R}^3)$ leads to the nonlocal boundary-value condition

$$\psi_s + (\psi, f) = 0. \quad (6)$$

The operator L_f defined in such a way for $f \in L_2(\mathbb{R}^3)$ will be self-adjoint on the space $L_2(\mathbb{R}^3)$. A more general result is proved in this paper on self-adjointness of an operator of the form (3) on $L_2(\mathbb{R}^3)$, if its domain is given in terms of nonlocal boundary-value conditions that generalize (2),

$$\psi_s + (\psi, f_2) = \alpha[\psi_r - (\psi, f_1)], \quad (7)$$

with a real parameter α and linearly dependent functions $f_1, f_2 \in L_2(\mathbb{R}^3)$. Such an operator will be called a Schrödinger operator with a *nonlocal point interaction*. The model (3), (7) belongs to the class of exactly solvable models [2,3]. It is sufficiently rich, since it includes not only the number parameter α but also the function parameters f_1, f_2 . This extends the scope of applications for models with point interactions.

2. A one-dimensional Schrödinger operator with nonlocal point interactions

We first introduce necessary notations and recall needed facts.

For a function ψ that has a discontinuity in a point $x = x_0$, we denote, respectively, by $\psi_r(x_0) = \frac{1}{2}[\psi(x_0 + 0) + \psi(x_0 - 0)]$ and $\psi_s(x_0) = \psi(x_0 + 0) - \psi(x_0 - 0)$ the mean value

and the jump of the function in the point $x = x_0$. For $x_0 = 0$, these numbers will be denoted by ψ_r and ψ_s , respectively.

For functions u, v in the Sobolev space $W_2^2(\mathbb{R}^1 \setminus \{0\})$, we have the following Green's formula:

$$(-u'', v)_{L_2} - (u, -v'')_{L_2} = (\Gamma_1 u, \Gamma_0 v)_{E^2} - (\Gamma_0 u, \Gamma_1 v)_{E^2}, \quad (8)$$

where the vectors $\Gamma_1 u = \text{col}(u'_s, u_s)$ and $\Gamma_0 u = \text{col}(u_r, -u'_r)$ define the boundary-value data for the function u [4,6]. Similar notations are used for the function v .

Let us consider, on the space $L_2(\mathbb{R}^1)$, a one-dimensional Schrödinger operator with a nonlocal point interaction in the point $x = 0$ of the form similar to (3),

$$L_f \psi = -\psi''(x) + f_1(x)\psi_r + f_2(x)\psi'_r, \quad x \neq 0, \quad (9)$$

with given functions $f_1, f_2 \in L_2(\mathbb{R}^1)$ and the maximal domain $D(L_f^{(\max)}) = W_2^2(\mathbb{R}^1 \setminus \{0\})$.

For functions $u, v \in W_2^2(\mathbb{R}^1 \setminus \{0\})$, the following Green's formula holds:

$$(L_f^{(\max)} u, v)_{L_2} - (u, L_f^{(\max)} v)_{L_2} = (\hat{\Gamma}_1 u, \hat{\Gamma}_0 v)_{E^2} - (\hat{\Gamma}_0 u, \hat{\Gamma}_1 v)_{E^2}, \quad (10)$$

where

$$\hat{\Gamma}_1 u = \text{col}(u'_s - (u, f_1), u_s + (u, f_2)), \quad \hat{\Gamma}_0 u = \text{col}(u_r, -u'_r). \quad (11)$$

The minimal operator $L_f^{(\min)}$ corresponding to (9) is the operator $-\frac{d^2}{dx^2}$ that is symmetric on $L_2(\mathbb{R}^1)$ and defined on functions that belong to the space $W_2^2(\mathbb{R}^1 \setminus \{0\})$ and satisfy the boundary-value conditions

$$\psi_r = 0, \quad \psi'_r = 0, \quad \psi_s + (\psi, f_2) = 0, \quad \psi'_s - (\psi, f_1) = 0. \quad (12)$$

For such a choice of the operator $L_f^{(\min)}$, we have

$$L_f^{(\max)} = (L_f^{(\min)})^*. \quad (13)$$

Theorem 1. Let $f_1, f_2 \in L_2(\mathbb{R}^1)$. Then Green's formula (10) defines a boundary-value triple $(E^2, \hat{\Gamma}_0, \hat{\Gamma}_1)$ for the operator $L_f^{(\min)}$.

Proof. This follows from the definition of a boundary-value triple (or the boundary-value space, BVS) [1–3,10,13,18]. Indeed, the boundary-value conditions (12) have the form $\hat{\Gamma}_0 \psi = 0$, $\hat{\Gamma}_1 \psi = 0$ and define the minimal operator. Green's formula (10) holds for the adjoint operator. The requirement that the mapping $(\hat{\Gamma}_0 \psi, \hat{\Gamma}_1 \psi) \mapsto E^4$ is surjective for $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ is a consequence of the following lemma. \square

Lemma 1. For any two vectors $F_0, F_1 \in E^2$ there is a function $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ such that $\hat{\Gamma}_0 \psi = F_0$ and $\hat{\Gamma}_1 \psi = F_1$.

Proof. Let vectors $F_0 = \text{col}(C_1, C_2)$ and $F_1 = \text{col}(C_3, C_4)$ be given. Let $\chi_\epsilon(x) \in C_0^\infty(\mathbb{R}^1)$, $0 < \epsilon < 1/2$, be a function such that $0 \leq \chi_\epsilon(x) \leq 1$ for any x , and $\chi_\epsilon(x) = 1$ for $|x| < \epsilon$ and $\chi_\epsilon(x) = 0$ for $|x| \geq 2\epsilon$. It is well known that such almost characteristic functions exist.

Consider a function $u_\epsilon(x) \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ that depends on four complex parameters C_1, C_2, B_1 , and B_2 ,

$$u_\epsilon(x) = \left[C_1 - C_2 x + \frac{1}{2}(B_1 + B_2 x) \text{sign } x \right] \chi_\epsilon(x). \quad (14)$$

It is seen from the construction that $\widehat{\Gamma}_0 u_\epsilon(x) = F_0 = \text{col}(C_1, C_2)$, $u_{\epsilon,s} = B_1$, and $u'_{\epsilon,s} = B_2$. Consider now a system of linear equations with respect to the unknowns B_1, B_2 with $C_i, i = 1, \dots, 4$, considered as given,

$$B_1 + (u_\epsilon, f_2) = C_4, \quad B_2 - (u_\epsilon, f_1) = C_3. \quad (15)$$

For ϵ small enough, the system (15) has a unique solution. Indeed, since $\|\chi_\epsilon\|_{L_2} \leq 2\sqrt{\epsilon}$, the determinant of system (15) approaches 1 as $\epsilon \rightarrow 0$, thus it is not zero if $\sqrt{\epsilon}(\|f_1\| + \|f_2\|) < \frac{1}{4}$. This means that if B_1 and B_2 in definition (14) of the function u_ϵ are chosen to be solutions of system (15), then $\widehat{\Gamma}_0 u_\epsilon = (C_1, C_2)$ and $\widehat{\Gamma}_1 u_\epsilon = (C_3, C_4)$. \square

Remark 1. By setting $f_1 \equiv f_2 \equiv 0$ in (7) and Theorem 1, we get the known result that $(E^2, \Gamma_0, \Gamma_1)$ in Green's formula (8) is a boundary-value triple for the one-dimensional Schrödinger operator with a one-point interaction in the point $x = 0$. In particular, this gives a description of all self-adjoint one-dimensional Schrödinger operators on $L_2(\mathbb{R}^1)$ with a one-point interaction in terms of one of the following boundary-value conditions for $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ [4,8,12,14,15,17,18,25]:

$$(i) \quad (I - U)\Gamma_1\psi = i(I + U)\Gamma_0\psi, \quad (16)$$

where U is a unitary matrix in the space E^2 ;

$$(ii) \quad A\Gamma_0\psi = B\Gamma_1\psi, \quad (17)$$

where the two arbitrary matrices A and B in the space E^2 have the property $AB^* = BA^*$, and the rank of the corresponding matrix $\|A, B\|$ equals 2, see [1,7,16,25]. In particular, if $B = I$, then A can be any symmetric matrix and the boundary-value conditions (17) take the form

$$\Gamma_1\psi = A\Gamma_0\psi. \quad (18)$$

(iii) The boundary-value vector $\Gamma\psi = (\Gamma_0\psi, \Gamma_1\psi)$ lies on the Lagrangian plane in the space E^4 relatively to the indefinite scalar product $\langle x, y \rangle = (x_1, y_2)_{E^2} - (x_2, y_1)_{E^2}$, where $x = (x_1, x_2), y = (y_1, y_2) \in E^4$. In other words, the boundary-value conditions are given by defining a Lagrangian plane, a subspace of E^4 which is maximal isotropic with respect to the scalar product $\langle \cdot, \cdot \rangle$.

Remark 2. Statements (i)–(iii) in Remark 1 hold true for an operator L_f of the form (9), if Γ_0 and Γ_1 are replaced by $\widehat{\Gamma}_0$ and $\widehat{\Gamma}_1$ from the definition (11). In particular, we have the following theorem.

Theorem 2. Let $f_1, f_2 \in L_2(\mathbb{R}^1)$. Then the operator $L_{f,A}$ of the form (9) defined on functions $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ and satisfying the boundary-value condition

$$\widehat{\Gamma}_1\psi = A\widehat{\Gamma}_0\psi \quad (19)$$

with a symmetric operator A on E^2 is self-adjoint on the space $L_2(\mathbb{R}^1)$.

Proof. The statement of the theorem follows from Theorem 1 and general facts of the theory of boundary-value triples [1,13,18]. However, we will give a direct proof, since similar proofs will be used in the sequel.

It follows from Green's formula (10) that the operator $L_{f,A}$ of the form (9) defined on functions in the space $W_2^2(\mathbb{R}^1 \setminus \{0\})$ and satisfying the boundary-value conditions (19) is symmetric. Indeed, the right-hand side of Green's formula (10) is identically zero, since the operator A is symmetric on the space E^2 . Let us construct the operator $L_{f,A}^*$. This operator is a restriction of the operator $L_f^{(\max)}$ to all the functions $u \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ that make the right-hand side of Green's formula (10) vanish for all the functions v satisfying the boundary-value condition (19). Since the vector $\text{col}(v_r, -v_r')$ can be arbitrary by Lemma 1 and $\text{col}(v_s' - (v, f_1), v_s + (v, f_2)) = A \text{col}(v_r, -v_r')$, we see that the right-hand side of (10) is zero if and only if u satisfies conditions (19). This shows that $L_{f,A}^* = L_{f,A}$. \square

Let us now consider Schrödinger operators with nonlocal point interactions, which are more general than the L_f given by (9):

$$L_{f,A}\psi = -\psi'' + \sum_{i=1}^2 \sum_{j=1}^4 a_{ij} f_i(x) \psi_j, \quad (20)$$

where the functions f_1, f_2 belong to $L_2(\mathbb{R}^1)$, the constant matrices $A_1 = \|a_{ij}\|_{i,j=1}^2$, $A_2 = \|a_{i,2+j}\|_{i,j=1}^2$ are considered as given, and the vector $(\psi_1, \psi_2, \psi_3, \psi_4) = (\psi_s', \psi_s, \psi_r, -\psi_r')$ is the boundary-value datum for the function $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$.

Define an operator $L_{f,A;B}$ by (20) on the functions $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ that satisfy the boundary-value conditions

$$\Gamma_1 u - A_2^*(u, f) = B[\Gamma_0 u + A_1^*(u, f)] \quad (21)$$

with a given operator B on the space E^2 . We used the following notations in conditions (21):

$$A_2^*(u, f) = \begin{pmatrix} \bar{a}_{13} & \bar{a}_{23} \\ \bar{a}_{14} & \bar{a}_{24} \end{pmatrix} \begin{pmatrix} (u, f_1)_{L_2} \\ (u, f_2)_{L_2} \end{pmatrix},$$

$$A_1^*(u, f) = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix} \begin{pmatrix} (u, f_1)_{L_2} \\ (u, f_2)_{L_2} \end{pmatrix}.$$

Theorem 3. Let $f_1, f_2 \in L_2(\mathbb{R}^1)$, and the matrices A_1 and A_2 satisfy the condition

$$A_1 A_2^* = A_2 A_1^*.$$

Let the matrix B be symmetric. Then the operator $L_{f,A;B}$ is self-adjoint on the space $L_2(\mathbb{R}^1)$.

Proof. The proof of this theorem follows closely that of Theorem 2. Let us only mention the analogue of Green's formula (10) for operators of the form (20):

$$(L_{f,A}^{(\max)} u, v)_{L_2} - (u, L_{f,A}^{(\max)} v)_{L_2} = (\tilde{\Gamma}_1 u, \tilde{\Gamma}_0 v)_{E^2} - (\tilde{\Gamma}_0 u, \tilde{\Gamma}_1 v)_{E^2},$$

where $\tilde{\Gamma}_1 u = \Gamma_1 u - A_2^*(u, f)$, $\tilde{\Gamma}_0 u = \Gamma_0 u + A_1^*(u, f)$, and $D(L_{f,A}^{(\max)}) = W_2^2(\mathbb{R}^1 \setminus \{0\})$. \square

3. Singular perturbations

It is well known that a one-dimensional Schrödinger operator on $L_2(\mathbb{R}^1)$ with a point interaction can be given not only in terms of boundary-value conditions of type (16)–(18) but also in terms of singular perturbations [2–4,21,22],

$$L = -\frac{d^2}{dx^2} + a_{11}\delta(\cdot, \delta) + a_{12}\delta(\cdot, \delta') + a_{21}\delta'(\cdot, \delta) + a_{22}\delta'(\cdot, \delta'), \quad (22)$$

where the numbers a_{11} and a_{22} are real, and $a_{21} = \bar{a}_{12}$. Here, the operator L is defined on the space $W_2^2(\mathbb{R}^1 \setminus \{0\})$, where the operator $-\frac{d^2}{dx^2}$ is understood in the sense of distributions, and Dirac's δ -function δ and its derivative δ' are defined on the space $W_2^2(\mathbb{R}^1 \setminus \{0\})$ by $(\psi, \delta) = \psi_r$, $(\psi, \delta') = -\psi'_r$. Conditions on the functions ψ that imply that $L\psi$ belongs to the space $L_2(\mathbb{R}^1)$ have the form of the boundary-value conditions $\Gamma_1\psi = A\Gamma_0\psi$, where the matrix $A = \|a_{ij}\|_{i,j=1}^2$ is self-adjoint.

The most general boundary-value conditions (17) for self-adjoint Schrödinger operators with a point interaction can also be obtained by using singular perturbations of the form

$$\begin{aligned} L\psi = & -\frac{d^2}{dx^2}\psi(x) + \delta(x)[a_{11}\psi_r - a_{12}\psi'_r + a_{13}\psi'_s + a_{14}\psi_s] \\ & + \delta'(x)[a_{21}\psi_r - a_{22}\psi'_r + a_{23}\psi'_s + a_{24}\psi_s]. \end{aligned}$$

Using singular perturbations, it is also possible to define Schrödinger operators with nonlocal point interactions. For example, the operator $L_{f,A}$ in Theorem 2, which is self-adjoint on $L_2(\mathbb{R}^1)$, in the case where $A = 0$ can be represented as a singularly perturbed operator of the form

$$L_{f,A} = -\frac{d^2}{dx^2} + f_1(x)(\cdot, \delta) - f_2(x)(\cdot, \delta') + \delta(\cdot, f_1) - \delta'(\cdot, f_2). \quad (23)$$

The condition $L_{f,A}\psi \in L_2(\mathbb{R}^1)$ leads to the boundary-value conditions $\Gamma_1\psi = (\psi, f)$ of the form (19), if $A = 0$.

To describe the operators $L_{f,A}$ for $A \neq 0$ in terms of singular perturbations, it is sufficient to consider singular perturbations as sums of singular perturbations given in (23) and (20).

We remark that the operators in (20) and (23) with singular perturbations are formally self-adjoint.

4. Examples of one-dimensional Schrödinger operators with nonlocal point interactions

On the space $L_2(\mathbb{R}^1)$, consider an operator of the form (23) with $f_2 \equiv 0$, i.e.,

$$L_q = -\frac{d^2}{dx^2} + q(x)(\cdot, \delta) + \delta(x)(\cdot, q), \quad (24)$$

where $q \in L_2(\mathbb{R}^1)$. This operator is defined on functions $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ that satisfy the non-local boundary-value conditions

$$\psi_s = 0, \quad \psi'_s = (\psi, q), \quad (25)$$

and acts as follows:

$$L_q\psi = -\psi''(x) + q(x)\psi(0). \quad (26)$$

The operator L_q is self-adjoint on the space $L_2(\mathbb{R}^1)$ by Theorem 2. Let us construct the resolvent of the operator L_q , i.e., explicitly find the operator $(L_q + k^2)^{-1}$, where $\operatorname{Re} k > 0$. To do this, we must find a solution of the equation $-\psi''(x) + k^2\psi(x) + q(x)\psi(0) = h(x)$ satisfying conditions (25). This solution has the form

$$\psi(x) = \int G_k(x-s)h(s)ds - \psi(0) \int G_k(x-s)q(s)ds - \psi'_s G_k(x), \quad (27)$$

where $G_k(x) = \frac{1}{2k}e^{-k|x|}$ is the kernel of the resolvent of the unperturbed operator L_0 . From (27) and the second condition in (25) we get a system of two equations for $\psi(0)$ and ψ'_s ,

$$\begin{aligned} \left(1 + \int G_k(s)q(s)ds\right)\psi(0) + \frac{1}{2k}\psi'_s &= \int G_k(s)h(s)ds, \\ \iint G_k(x-s)q(s)\overline{q(x)}dsdx \cdot \psi(0) + \left(1 + \int G_k(s)\overline{q(s)}ds\right)\psi'_s \\ &= \iint G_k(x-s)h(s)\overline{h(x)}dsdx. \end{aligned} \quad (28)$$

The determinant of system (28) has the form

$$\begin{aligned} D(k) &= \left(1 + \int G_k(s)q(s)ds\right) \cdot \left(1 + \int G_k(s)\overline{q(s)}ds\right) \\ &\quad - \frac{1}{2k} \iint G_k(x-s)q(s)\overline{q(x)}dsdx. \end{aligned} \quad (29)$$

By substituting the solution of system (28) into the solution $\psi(x)$ given by (27), we obtain an explicit form for the kernel $\widehat{G}_k(x, s)$ of the integral operator $(L_q + k^2)^{-1}$,

$$\widehat{G}_k(x, s) = G_k(x-s) + \int G_k(x-\xi)q(\xi)d\xi \cdot E_1(s) + G_k(x) \cdot E_2(s), \quad (30)$$

where the functions E_1 and E_2 can be expressed by linear relations in terms of G_k and $G_k * \bar{q} = \int G_k(s-\xi)\bar{q}(\xi)d\xi$,

$$\begin{aligned} E_1(s) &= \frac{1}{2kD}(G_k * \bar{q})(s) - \frac{1 + (G_k * \bar{q})(0)}{D}G_k(s), \\ E_2(s) &= \frac{(G_k * q, q)}{D}G_k(s) - \frac{1 + (G_k * \bar{q})(0)}{D}(G_k * \bar{q})(s). \end{aligned} \quad (31)$$

Note that the determinant $D(k)$ in (29) depends on the spectral parameter k . Zeros of the equation $D(k) = 0$, for $k > 0$, define negative eigenvalues $\lambda = -k^2$ of the operator L_q .

Example 1. Let $q(x) = \alpha e^{-\mu|x|}$, $\mu > 0$. Then the characteristic equation $D(k) = 0$ has the form

$$k(k+\mu)^2 + 2\alpha(k+\mu) - \frac{\alpha^2}{2\mu} = 0. \quad (32)$$

The cubic polynomial (32) has a positive solution $k > 0$ for a negative α or for $\alpha > 4\mu^2$. In these cases, the left-hand side of (32) becomes negative if $k = 0$ and approaches $+\infty$ as $k \rightarrow +\infty$. Hence Eq. (32) has a solution on the positive semiaxis. Thus the operator L_q has a negative eigenvalue for $\alpha < 0$ and $\alpha > 4\mu^2$. The corresponding eigenfunction is

$$\psi_k(x) = \int G_k(x-s)q(s)ds - e^{-k|x|} \left(1 + \int G_k(s)q(s)ds\right).$$

Example 2. Let us again consider the operator in Example 1 with $\mu = 1$,

$$L_\alpha \psi = -\psi'' + \alpha e^{-|x|} \psi_r. \quad (33)$$

We will consider the operator L_α on the Hilbert space $W_2^1(\mathbb{R}^1)$ with the domain $D(L_\alpha) = W_2^3(\mathbb{R}^1)$ [5,23]. Since $\psi_r = \psi(0) = (\psi, \frac{1}{2}e^{-|x|})_{W_2^1}$, the operator L_α is a one-dimensional regular symmetric perturbation of the operator $L_0 = -\frac{d^2}{dx^2}$ that is self-adjoint on the space $W_2^1(\mathbb{R}^1)$,

$$L_\alpha = L_0 + 2\alpha g(x)(\cdot, g)_{W_2^1}, \quad (34)$$

where $g(x) = \frac{1}{2}e^{-|x|}$.

Hence, the operator L_α is self-adjoint on the space $W_2^1(\mathbb{R}^1)$ and its resolvent $(L_\alpha + k^2)^{-1}$ is an integral operator with the kernel

$$\widehat{G}_\alpha(x, s) = G_k(x - s) - \frac{\alpha}{k^2 + k + \alpha} E_k(x) G_k(s), \quad (35)$$

where $G_k(x) = \frac{1}{2k}e^{-k|x|}$ and $E_k(x) = \frac{k}{k-1}(e^{-|x|} - \frac{1}{k}e^{-k|x|})$.

The resolvent (35) has a pole in k , if k is a solution of the characteristic equation

$$k^2 + k + \alpha = 0. \quad (36)$$

If α is negative, Eq. (36) has a positive root, $k_0 = 2|\alpha|(1 + \sqrt{1 + 4|\alpha|})^{-1}$, and, consequently, for a negative α , the operator L_α , which is self-adjoint on the space $W_2^1(\mathbb{R}^1)$, has a negative eigenvalue, $\lambda_0 = -k_0^2$, with an eigenfunction $E_{k_0}(x)$.

Note that for real k , the kernel $\widehat{G}_\alpha(x, s)$ of the operator $(L_\alpha + k^2)^{-1}$ is not Hermitian, although the operator is self-adjoint on $W_2^1(\mathbb{R}^1)$. An integral operator with the kernel $\widehat{G}_\alpha(x, s)$ is self-adjoint on the space $W_2^1(\mathbb{R}^1)$, since this kernel, by (35), can be represented as a sum of a Hermitian kernel, which depends on the difference of the arguments, and the degenerate kernel $K(x, s) = C_1 E_k(x) G_k(s)$ satisfying the condition $(-\frac{d^2}{dx^2} + 1)E_k(x) = C_2 \overline{G_k(x)}$, where C_i are real constants. These conditions are sufficient to imply that the integral operator is self-adjoint on the space $W_2^1(\mathbb{R}^1)$.

Comparing the results in Examples 1 and 2 we see that the spectral properties of the operator L_f essentially depend on the boundary-value conditions and the space on which the operator is considered.

5. Scattering problem for one-dimensional Schrödinger operator with nonlocal point interaction

Consider the scattering problem for a Schrödinger operator L_q of the form (24) with a non-local potential $q \in L_2(\mathbb{R}^1)$. Since such a model is exactly solvable, the scattering operator can be found explicitly in terms of the potential q . Indeed, consider the eigenfunctions of the operator L_q , which correspond to points of the continuous spectrum and such that the scattering operator is expressed in terms of the asymptotics, as $x \rightarrow \pm\infty$, of these eigenfunctions [8,11,19]. Using (26), consider the equation

$$-\psi''(x) + q(x)\psi(0) = \lambda^2 \psi(x) \quad (37)$$

together with the boundary-value conditions (25).

If $\lambda \in \mathbb{R}^1$, Eq. (37) has bounded solutions $\psi_+(x; \lambda)$ defined on the positive half-axis, $x \geq 0$, and $\psi_-(x; \lambda)$ defined on the negative half-axis, $x \leq 0$. They are given by

$$\begin{aligned}\psi_+(x; \lambda) &= e^{i\lambda x} + (\mathcal{G}_\lambda q_+)(x), \quad x \geq 0, \\ \psi_-(x; \lambda) &= e^{i\lambda x} + (\mathcal{G}_\lambda q_-)(x), \quad x \leq 0,\end{aligned}\quad (38)$$

where

$$\begin{aligned}(\mathcal{G}_\lambda q_+)(x) &= -\frac{1}{\lambda} \left(\int_0^x e^{i\lambda s} \sin \lambda s q(s) ds + \int_x^\infty \sin \lambda x e^{i\lambda s} q(s) ds \right), \\ (\mathcal{G}_\lambda q_-)(x) &= \frac{1}{\lambda} \left(\int_x^0 e^{i\lambda s} \sin \lambda s q(s) ds + \int_{-\infty}^x \sin \lambda x e^{i\lambda s} q(s) ds \right).\end{aligned}\quad (39)$$

These solutions play the role of the usual Jost solutions for one-dimensional Schrödinger operator with a regular potential [11,19,24].

It follows from (38) that

$$\begin{aligned}\psi_+(0; \lambda) &= 1, \quad \psi_+(x; \lambda) = e^{i\lambda x} \left[1 - \frac{1}{\lambda} q_{+,s}(\lambda) \right] + o(1), \quad x \rightarrow +\infty, \\ \psi_-(0; \lambda) &= 1, \quad \psi_-(x; \lambda) = e^{i\lambda x} \left[1 + \frac{1}{\lambda} q_{-,s}(\lambda) \right] + o(1), \quad x \rightarrow -\infty,\end{aligned}\quad (40)$$

where $q_{\pm,s}(\lambda)$ are sin-Fourier transforms of the function q restricted to the positive half-axis, q_+ , and the negative half-axis, q_- , respectively,

$$q_{+,s}(\lambda) = \int_0^\infty \sin \lambda x q(x) dx, \quad q_{-,s}(\lambda) = \int_{-\infty}^0 \sin \lambda x q(x) dx. \quad (41)$$

Consider now the general solution of Eq. (37) taken as a superposition of the solutions ψ_\pm ,

$$\psi(x) = \begin{cases} a_1 \psi_+(x; \lambda) + b_2 \psi_+(x; -\lambda), & x \geq 0, \\ b_1 \psi_-(x; \lambda) + a_2 \psi_-(x; -\lambda), & x \leq 0, \end{cases} \quad (42)$$

where the constants a_1, a_2, b_1, b_2 are chosen such that the solution ψ satisfies the boundary-value conditions (25). These constants must satisfy the following system:

$$\begin{aligned}a_1 + b_2 - a_2 + b_1 &= 0, \\ a_1[i\lambda - \tilde{q}_+(\lambda)] + b_2[-i\lambda - \tilde{q}_+(-\lambda)] - a_2[-i\lambda + \tilde{q}_-(-\lambda)] - b_1[i\lambda + \tilde{q}_-(\lambda)] \\ &= a_1(\psi_+(\cdot; \lambda), q_+(\cdot))_{L_2(\mathbb{R}_+)} + b_2(\psi_+(\cdot; -\lambda), q_+(\cdot))_{L_2(\mathbb{R}_+)} \\ &\quad + a_2(\psi_-(\cdot; \lambda), q_-(\cdot))_{L_2(\mathbb{R}_-)} + b_1(\psi_-(\cdot; -\lambda), q_-(\cdot))_{L_2(\mathbb{R}_-)},\end{aligned}\quad (43)$$

where $\tilde{q}_+(\lambda) = \int_0^\infty e^{i\lambda s} q(s) ds$, $\tilde{q}_-(\lambda) = \int_{-\infty}^0 e^{i\lambda s} q(s) ds$ are the Fourier transforms of the restrictions of the function q to the positive and the negative half-axes, q_+ and q_- , respectively.

Using asymptotics (40) we come to the conclusion that the solution (42) is the sum of the incident waves, which come from the right and the left and have the corresponding amplitudes $A_1 = a_1[1 - \frac{1}{\lambda} q_{+,s}(\lambda)]$ and $A_2 = a_2[1 + \frac{1}{\lambda} q_{-,s}(\lambda)]$, and the scattered waves, which are scattered to the left and to the right and have the respective amplitudes $B_1 = b_1[1 + \frac{1}{\lambda} q_{-,s}(\lambda)]$ and

$B_2 = b_2[1 - \frac{1}{\lambda}q_{+,s}]$. If the amplitudes of the incident waves are given, $A = \text{col}(A_1, A_2)$, then system (43) permits to uniquely find the amplitudes of the scattered waves, $B = \text{col}(B_1, B_2)$. The scattering operator $S(\lambda)$, for the Schrödinger operator L_q , transforms the vector A into the vector B ,

$$B = S(\lambda)A. \quad (44)$$

The scattering operator $S(\lambda)$ is a (2×2) -matrix-valued function of the spectral parameter $\lambda \in \mathbb{R}^1$. Elements of the scattering matrix $S(\lambda) = \|S_{ij}(\lambda)\|_{i,j=1}^2$ have the meaning of the transient coefficients, S_{11} is from the right and S_{22} is from the left, and the reflection coefficients, S_{12} is to the right and S_{21} is to the left. Using system (43) one can easily obtain an explicit expression for the matrix elements of the scattering matrix in terms of the nonlocal potential. In particular, for real q , $q = \bar{q}$, we have

$$\begin{aligned} S_{22} &= S_{11}, & 1 + S_{12} &= S_{11}Q, & 1 + S_{21} &= S_{11}Q^{-1}, \\ \text{Re}(S_{11}^{-1}) &= \frac{1}{2}(Q + Q^{-1}), & Q &= \frac{1 + q_{-,s}(\lambda)}{1 - q_{+,s}(\lambda)}, \\ \text{Im}(S_{11}^{-1}) &= -\frac{\lambda}{(\lambda - q_{+,s}(\lambda))(\lambda + q_{-,s}(\lambda))} \left[q_{+,c}(\lambda) + q_{-,c}(\lambda) \right. \\ &\quad \left. - \frac{1}{\pi} \int_0^\infty \frac{q_{+,s}^2(\mu) + q_{-,s}^2(\mu)}{\mu^2 - \lambda^2} d\mu \right], \end{aligned} \quad (45)$$

where the integral is understood in the sense of Cauchy is principal value; $q_{+,c}(\lambda) = \int_0^\infty \cos \lambda x q(x) dx$ and $q_{-,c} = \int_{-\infty}^0 \cos \lambda x q(x) dx$.

Formulas (45) can be used for solving the inverse scattering problem, namely the problem for determining the nonlocal potential q from a known scattering operator $S(\lambda)$, $\lambda \in \mathbb{R}^1$. This is the case, e.g., if the nonlocal potential is an odd function, $q(x) = -q(-x)$. Then $q_{-,s}(\lambda) = q_{+,s}(\lambda)$ and formulas (45) give an explicit expression for the sin-Fourier transform of the function q ,

$$q_{+,s}(\lambda) = \lambda \frac{1 - S_{11} + S_{12}}{1 + S_{11} + S_{12}} = -\lambda \frac{1 - S_{11} + S_{21}}{1 + S_{11} + S_{21}}. \quad (46)$$

The nonlocal potential q can now be recovered by using the inverse Fourier transform,

$$q(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x q_{+,s}(\lambda) d\lambda. \quad (47)$$

If the nonlocal potential is an even real-valued function, $q(x) = q(-x) = \overline{q(x)}$, then we have that $Q = 1$. Using the identity $\text{Re } S_{11}^{-1} = \frac{1}{2}(Q + Q^{-1})$, we see that $\text{Re } S_{11}^{-1} = 1$ for $\lambda \in \mathbb{R}^1$ is a necessary and sufficient condition for the nonlocal potential to be even. In this case we can get from (45) the linear equation for the function $u(\lambda) = q_{+,s}^2(\lambda) - 2\lambda q_{+,s}(\lambda)$,

$$\frac{2\lambda}{\pi} \int_0^\infty \frac{u(\mu)}{\mu^2 - \lambda^2} d\mu = a(\lambda)(\lambda^2 + u(\lambda)), \quad (48)$$

where $a(\lambda) = \text{Im}(S_{11}^{-1}(\lambda))$. The inverse scattering problem for even nonlocal potentials is solved by Eq. (48) with given function $\text{Im}(S_{11}^{-1})$ if one can assume a priori, for example, that

$\int_0^\infty |xq_+(x)| dx < 1$. In this case we have $|q_{+,s}(\lambda)| < |\lambda|$ and the function $q_{+,s}$ is uniquely defined from the quadratic equation $q_{+,s}^2(\lambda) - 2\lambda q_{+,s}(\lambda) = u(\lambda)$ by the function u .

The inverse scattering problem for a general nonlocal potential needs a special study.

6. A Schrödinger operator with nonlocal point interactions in the three-dimensional space

The Schrödinger operator $-\Delta_{\alpha,0}$ with a one-point interaction in the point $x = 0$, self-adjoint on the space $L_2(\mathbb{R}^3)$, is a self-adjoint extension of the minimal operator $L_{\min} = -\Delta|_{C_0^\infty(\mathbb{R}^3 \setminus \{0\})}$ [2,3,9,20]. The domain of the adjoint operator L_{\min}^* is the direct sum of the Sobolev space $W_2^2(\mathbb{R}^3)$ and the one-dimensional space of (equivalence classes of) functions that are infinitely differentiable for $x \neq 0$ and have a singularity in $x = 0$ of the type $|x|^{-1}$, for example, all the functions that are multiples of $\frac{e^{-|x|}}{|x|}$,

$$D(L_{\min}^*) = W_2^2(\mathbb{R}^3) \dot{+} \left\{ \frac{e^{-|x|}}{|x|} \right\}. \quad (49)$$

Here $L_{\min}^* \psi(x) = -\Delta \psi(x)$ for $x \neq 0$.

For functions $\psi \in D(L_{\min}^*)$, there are the numbers ψ_s and ψ_r given by formula (4). The following Green's formula holds:

$$(L_{\min}^* u, v) - (u, L_{\min}^* v) = u_r \bar{v}_s - u_s \bar{v}_r. \quad (50)$$

Green's formula (50) defines a boundary-value triple $(E^1, \Gamma_0, \Gamma_1)$ for the operator L_{\min} , where $\Gamma_0 \psi = -\psi_r$ and $\Gamma_1 \psi = \psi_s$. This gives a description of all self-adjoint extensions of the operator L_{\min} with the boundary-value conditions (2).

Consider now an operator of the form (3) with functions f_1, f_2 that are linearly dependent, that is, the operator

$$L_f \psi = -\Delta \psi(x) + f(x)(a\psi_r + b\psi_s), \quad (51)$$

where a and b are real numbers. This operator is defined on the space $D(L_{\min}^*)$ given by (49), and we have $L_f \psi = L_{\min}^* \psi + f(x)(a\psi_r + b\psi_s)$. Operator L_f satisfies Green's formula,

$$(L_f u, v) - (u, L_f v) = \Gamma_1 u \cdot \overline{\Gamma_0 v} - \Gamma_0 u \cdot \overline{\Gamma_1 v}, \quad (52)$$

where $\Gamma_1 u = u_s + a(u, f)$, $\Gamma_0 u = -u_r + b(u, f)$.

In the same way as in the proof of Lemma 1, one can show that for any two complex numbers C_1 and C_2 there is a function $u \in D(L_{\min}^*)$ such that $\Gamma_0 u = C_1$ and $\Gamma_1 u = C_2$.

Theorem 4. Let $f \in L_2(\mathbb{R}^3)$ in (51), and let the numbers a, b be real. Then the operator $L_{f,\alpha}$ defined on all functions in the space (49) and satisfying the boundary-value conditions

$$u_s + a(u, f) = \alpha[u_r - b(u, f)] \quad (53)$$

with a real number α is self-adjoint on the space $L_2(\mathbb{R}^3)$.

Proof. The proof is conducted similarly to that of Theorem 2 and follows from the fact that Green's formula (52) defines a boundary-value triple $(E^1, \Gamma_0, \Gamma_1)$ for the operator L_f^* . \square

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